# Pseudo-Hermitian Interactions in Dirac Theory: Examples

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**Abstract**

We consider a couple of examples to study the pseudo-Hermitian interaction in relativistic quantum mechanics. Rasbha interaction, commonly used to study the spin Hall effect, is considered with imaginary coupling. The corresponding Dirac Hamiltonian is shown to be parity pseudo-Hermitian. In the other example we consider parity pseudo-Hermitian scalar interaction with arbitrary parameter in Dirac theory. In both the cases we show that the energy spectrum is real and all the other features of non-relativistic pseudo-Hermitian formulation are present. Using the spectral method the positive definite metric operator ($\eta$) has been calculated explicitly for both the models to ensure positive definite norms for the state vectors.

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# 1 Introduction

Non-Hermitian quantum mechanics has recently created a lot of interest. This is due to the observation by Bender and Boettcher [1, 2] that with properly defined boundary conditions, the spectrum of the system described by the Hamiltonian $H = p^2 + x^2(ix)^\nu,\ \ \nu \geq 0$ is real positive and discrete. The reality of the spectrum is a consequence of unbroken PT (combined Parity [P] & Time reversal [T]) invariance of the Hamiltonian, $[H, PT]\psi = 0$ and the spectrum becomes partially complex when the PT symmetry is broken spontaneously [3, 4].

This new result has given rise to growing interest in the literature, see for examples, [3]-[31]. Past few years many non-Hermitian but PT symmetric systems have been investigated including (i) quantum mechanics of single particle in one space dimension [1]-[16], (ii) exactly solvable many particle quantum systems in one space dimension [17]-[22], (iii) field theoretic models [23]-[27]. To develop a consistent quantum theory for these systems, one encounters severe difficulties [28, 29]. Particularly, the eigenstates of PT symmetric non-Hermitian Hamiltonians, with real eigenvalues only, do not satisfy standard completeness relations. More importantly the eigenstates have negative norms if one takes the natural inner product associated with such systems defined as

$$< f \mid g >_{PT} = \int d^3x [PTf(x)] g(x). \tag{1.1}$$

These problems are overcome by introducing a new symmetry [C], analogous to charge conjugation symmetry, associated with all such systems with equal number of negative and positive norm states [30, 31]. This allows to introduce an inner product structure associated with CPT conjugate as

$$< f \mid g >_{CPT} = \int d^3x [CPTf(x)] g(x), \tag{1.2}$$

for which the norms of the quantum states are positive definite and one gets usual completeness relation. As a result, the Hamiltonian and its eigenstates can be extended to complex domain so that the associated eigenvalues are real and underlying dynamics is unitary. Thus we have a fully consistent quantum theory for non-Hermitian but PT invariant systems.

In an alternative approach [32]-[38], it has been shown that the reality of the spectrum of a non-Hermitian system is due to so called pseudo-Hermiticity properties of the Hamiltonian. A Hamiltonian is called $\eta$ pseudo-Hermitian if it satisfies the relation

$$\eta H \eta^{-1} = H^{\dagger}, \tag{1.3}$$

where $\eta$ is some linear Hermitian and invertible operator. All PT symmetric non-Hermitian systems are pseudo-Hermitian where parity operator plays the role of $\eta$. However there are many pseudo-Hermitian systems with completely real spectrum which are not PT symmetric [39]-[41]. In the formulation of pseudo-Hermitian quantum mechanics, the scalar product is defined as

$$< f \mid g >_{\eta} = < f \mid \eta g > = \int d^3x f^* \eta g, \tag{1.4}$$

to get rid of the conceptual difficulties arise in such theories. Equation (1.2) which defines the scalar product in case of non-Hermitian PT invariant theory is a special case of Eq. (1.4) [42].

If the Hamiltonian is diagonalizable and has a real and discrete spectrum, it is always possible to find a positive definite $\eta$ [32, 43] such that the norms of all the eigenstates become positive definite and they satisfy usual completeness relation. Thus one can have completely consistent quantum theory for the pseudo-Hermitian systems provided atleast one positive definite $\eta$ is contructed.

However, most of the previous works in the pseudo-Hermitian quantum mechanics have been carried out in the non-relativistic framework [3]. The purpose of this paper is to extend the results of pseudo-Hermitian quantum mechanics for relativistic systems. Here we consider a couple of examples of pseudo-Hermitian Hamiltonian and show that the energy spectrum obtained by solving corresponding Dirac equation is also real. In the first example, we study the famous Rashba interaction [4] [48] with imaginary coupling which

[3]Some attempts to study relativistic PT invariant non-Hermitian system can be found in Refs. [43]-[47].

[4]This type of spin orbit coupling is widely used to study spin Hall effect [49, 50, 51, 52]. In the simplest version of spin Hall effect, an electric current passes through a sample with spin orbit interaction and induces a spin polarization near the lateral edges, with opposite polarization at opposite edges. This effect does not require an external magnetic field or magnetic order in the equilibrium state before the current is applied.

is pseudo-Hermitian with respect to parity operator. We consider a Dirac particle in 1+1 dimension, moving under arbitrary scalar pseudo-Hermitian interaction in other example. The norms of the state vectors, defined according to the modified rule of scalar product [ i.e. Eq. (1.4)] will be positive definite if we find a positive $\eta$ for the above two models. We have constructed positive definite $\eta$ for both the models using spectral method [35, 36].

This paper is arranged as follows. In section II, we consider the motion of a Dirac particle in the x-y plane experiencing Rashba interaction with imaginary coupling. The pseudo-Hermitian scalar interaction of Dirac particle in 1+1 dimension is discussed in section III. Positive definite $\eta$ for both the models have been constructed explicitly in section IV. Last section reveals for conclusion and summary.

# 2 Dirac equation with Rashba interaction

We consider the motion of a particle in the x-y plane described by the Hamiltonian

$$H = c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m_0 c^2 + \lambda_1(\boldsymbol{\sigma}\times\mathbf{p})\cdot\hat{\mathbf{z}}, \tag{2.1}$$

where the last term (with real $\lambda_1$) is known as the Rashba interaction term. This type of interaction is widely used to discuss spin Hall effect [49, 50, 51, 52] in non-relativistic formulation. However, the Hamiltonian is no longer Hermitian if we consider the $\lambda_1$ is complex. Further it can be shown that this Hamiltonian is a parity pseudo-Hermitian when $\lambda_1(\equiv i\lambda)$ is purely imaginary.

$$\begin{aligned} H^\dagger &= c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m_0 c^2 - i\lambda(\boldsymbol{\sigma}\times\mathbf{p})\cdot\hat{\mathbf{z}} \\ &\neq H \\ &= PHP^{-1}, \end{aligned} \tag{2.2}$$

where $P(=\beta e^{i\delta})$ is parity operator in Dirac theory. $(\boldsymbol{\sigma}\times\mathbf{p})\cdot\hat{\mathbf{z}}$ changes sign under parity transformation

$$P(\boldsymbol{\sigma}\times\mathbf{p})\cdot\hat{\mathbf{z}}P^{-1} = -(\boldsymbol{\sigma}\times\mathbf{p})\cdot\hat{\mathbf{z}},$$

as under parity:

$\boldsymbol{\sigma}\rightarrow\boldsymbol{\sigma}$ , $\mathbf{p}\rightarrow-\mathbf{p}$, $\hat{\mathbf{z}}\rightarrow-\hat{\mathbf{z}}$ and $\beta$ anti-commutes with $\sigma_x, \sigma_y$.

Now we solve the Dirac equation $H\psi = E\psi$ for this system to find the energy eigenvalues. To do that we consider the wave function in terms of

components, $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ and restrict ourselves to 2-dimension only. Further, we consider the particular representation of Dirac matrices in 2-dimension as, $\alpha = \sigma, \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $\sigma_{\mathbf{i}}$[ for $i = x, y, z$] are Pauli spin matrices. Then the Hamiltonian in equation (2.1) can be written as

$$\begin{aligned} H &= c\sigma_x p_x + c\sigma_y p_y + \beta m_0 c^2 + i\lambda(\sigma_x p_y - \sigma_y p_x) \\ &= \begin{bmatrix} m_0 c^2 & (c-\lambda)p_- \\ (c+\lambda)p_+ & -m_0 c^2 \end{bmatrix}, \end{aligned} \tag{2.3}$$

where $p_\pm = p_x \pm ip_y$. The Dirac equation $H\psi = E\psi$ can be written as pair of coupled equations of the components $\psi$ and $\chi$ as

$$\begin{aligned} (c-\lambda)p_-\chi &= (E - m_0c^2)\phi, \\ (c+\lambda)p_+\phi &= (E + m_0c^2)\chi. \end{aligned} \tag{2.4}$$

On eliminating one component $\chi$ in terms of others, we obtain the equation

$$p_- p_+ \phi = \epsilon\phi, \tag{2.5}$$

where $\epsilon = \frac{E^2 - m_0^2c^4}{c^2-\lambda^2}$. Equation (2.5) can be interpreted as Schrodinger equation for a free particle moving in the x-y plane. The corresponding energy eigenvalues $\epsilon$ are given as $\hbar^2(k_x^2 + k_y^2)$, which are real and positive. Therefore the energy eigenvalues E for the Dirac Hamiltonian for the pseudo-Hermitian system, described in Eq.(2.1), are given by

$$E = \pm\sqrt{m_0^2c^4 + (c^2 - \lambda^2)\hbar^2(k_x^2 + k_y^2)}. \tag{2.6}$$

Thus the energy spectrum for a pseudo-Hermitian [ Rashba interaction with imaginary coupling ] relativistic system is completely real for $\lambda^2 < c^2$. Same conclusion can be drawn by eliminating other component $\phi$ from the Eqs. in (2.4).

# 3 Dirac equation with scalar pseudo-Hermitian interaction

Let us consider a Dirac particle of rest mass $m_0$ subjected to a scalar pseudo-Hermitian potential $V_s$. The dynamics of such a system can be described by

the Hamiltonian

$$H = c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m_0 c^2 + V_s. \tag{3.1}$$

For simplicity, we restrict ourselves to one space dimension and we choose the scalar pseudo-Hermitian potential as

$$V_s = V(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.2}$$

where $V(x)$ is an arbitrary real function and assumed to have even parity, $V(-x) = V(x)$, for later convenience. We note that the Hamiltonian given in Eq. (3.1) is not Hermitian as $V_s^\dagger = -V_s \neq V_s$, but it is parity pseudo-Hermitian, i.e.

$$H^\dagger = PHP^{-1}, \tag{3.3}$$

where $P$ is the parity operator and in relativistic formulation is given by $P = \beta e^{i\delta}$. This is so because, $c\alpha \cdot \mathbf{p} + \beta m_0 c^2$ is a parity invariant term and

$$PV_sP^{-1} = \beta e^{i\delta} V(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \beta^{-1} e^{-i\delta} = -V_s = V_s^\dagger,$$

where we have chosen a particular representation of Dirac matrices $\alpha$ and $\beta$ in 1+1 dimension as

$$\alpha_x = \sigma_x \text{ and } \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\sigma_x$ is the first of Pauli's $2 \times 2$ spin matrices. 1+1 dimensional Dirac equation for the system can be written as

$$\left[c\sigma_x p_x + \beta m_0 c^2 + V_s\right] \psi = E\psi. \tag{3.4}$$

By considering $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, the above equation can be written in component form

$$\left[V(x) - ic\hbar \frac{d}{dx}\right] \chi = (E - m_0 c^2)\phi, \tag{3.5}$$

$$\left[-V(x) - ic\hbar \frac{d}{dx}\right] \phi = (E + m_0 c^2)\chi. \tag{3.6}$$

By eliminating $\chi$ from the above coupled differential equations, we obtain

$$\left[-\hbar^2 \frac{d^2}{dx^2} + U\right] \phi = \epsilon\phi, \tag{3.7}$$

where

$$U = -i\hbar\frac{dV}{dx} - V^2 \text{ and } \epsilon = E^2 - m_0^2 c^4.$$

Eq. (3.7) is nothing but non-relativistic Schrodinger equation for a particle of mass $m = \frac{1}{2}$ subjected to a complex potential $U$ and $\epsilon$ is the energy eigenvalues for the particle. Even though the potential U is complex, remarkably it is parity pseudo-Hermitian as

$$U^* = \tilde{P} U \tilde{P}^{-1}.$$

where $\tilde{P}$ is parity operator in non-relativistic quantum theory. Note we have assumed $V(-x) = V(x)$. Hence the energy eigenvalues,$\epsilon$ for this effective non-relativistic theory is real or occure in complex conjugate pairs[33]depending on whether the symmetry is broken or not. . The energy eigenvalues of the relativistic particle is given in terms of $\epsilon$ by

$$E = \pm\sqrt{\epsilon + m_0^2 c^4}. \tag{3.8}$$

Now if $\epsilon$ is real and positive, the whole spectrum of the relativistic particle is real without any restriction. On the other hand, if $\epsilon$ is real but negative i.e. $\epsilon = -|\epsilon|$, then also the spectrum is real provided $|\epsilon| < m_0^2 c^4$. However for arbitrary $V(x)$ energy eigenvalues are not real always as expected. Exactly similar conclusion can be drawn by eliminating $\phi$ from the equations (3.5) and (3.10).

Now we consider a special case when, $V(x) = V_0$, independent of x

$$V_s = V_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.9}$$

where $V_0$ is an arbitrary real constant. The Dirac equation in Eq. **??**3.1) can be written as

$$\begin{bmatrix} m_0 c^2 & cp_x + V_0 \\ cp_x - V_0 & -m_0 c^2 \end{bmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = E \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{3.10}$$

and in terms of components,

$$\begin{aligned} (cp_x + V_0)\chi &= (E - m_0c^2)\phi, \\ (cp_x - V_0)\phi &= (E + m_0c^2)\chi. \end{aligned} \tag{3.11}$$

By solving the above coupled differential equations, we obtain the energy eigenvalues as

$$E = \pm\sqrt{\hbar^2 c^2 k_x^2 + m_0^2 c^4 - V_0^2} \tag{3.12}$$

with

$$p_x^2 \phi = \hbar^2 k_x^2 \phi \tag{3.13}$$

The energy eigenvalues are always real for sufficiently weak pseudo-Hermitian interaction.

# 4 Construction of positive definite $\eta$

We have shown in both the examples that even in relativistic quantum mechanics we can have real eigenvalues if the relativistic Hamiltonian is parity pseudo-Hermitian. In order to have a consistent formulation for these systems we need to construct a positive definite $\eta$ such that inner product is well defined. In this section we intend to construct the positive definite $\eta$ explicitly by using the spectral method as defined in Refs. [35, 36]. The positive definite $\eta$ for a $\eta$-pseudo Hermitian theory described by the Hamiltonian $H$ is defined as [35, 36],

$$\eta = \sum_{i=1}^{2} |u_i >< u_i|, \tag{3.14}$$

where $|u_i>; \quad i = 1, 2$ are the Dirac spinors associated with $H^\dagger (= \eta H \eta^{-1})$. Following this method we construct the positive definite $\eta$ for both the models discussed above.

**Case I: Pseudo-Hermitian Rasbha Interaction:**

In this case, $H^\dagger$ can be written from Eq. (2.3) as

$$H^\dagger = \left[ \begin{array}{cc} m_0c^2 & (c+\lambda)\mathbf{p}_- \\ (c-\lambda)\mathbf{p}_+ & -m_0c^2 \end{array} \right], \tag{3.15}$$

and let $\left(\begin{array}{c} v_1 \\ v_2 \end{array}\right)$ be the two component spinor for $H^\dagger$ corresponding to the energy E, and satisfy,

$$H^\dagger \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right) = E \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right) \tag{3.16}$$

The spinors are calculated as

$$|u_1 >= \left(\begin{array}{c} 1 \\ \frac{(c-\lambda)p_+}{E+m_0c^2} \end{array}\right); \text{and } |u_2 >= \left(\begin{array}{c} \frac{-(c+\lambda)p_-}{E+m_0c^2} \\ 1 \end{array}\right) \tag{3.17}$$

when $E$ is given in Eq. (2.6) and $p_+, p_-$ are eigenvalues of the operator $p_+$ and $p_-$ respectively. Substituting Eq. (3.17) in Eq. (3.14) we obtain the positive definite $\eta$, as

$$\eta_I = \left[\begin{array}{cc} 1 + \frac{(c+\lambda)^2}{(E-m_0c^2)^2} p_+p_- & \frac{2p_-(cE+\lambda m_0c^2)}{E^2-m_0^2c^4} \\ \frac{2p_+(cE+\lambda m_0c^2)}{E^2-m_0^2c^4} & 1 + \frac{(c-\lambda)^2}{(E+m_0c^2)^2} p_+p_- \end{array}\right] \tag{3.18}$$

**Case II: Pseudo-Hermitian scalar interaction**

It is difficult to obtain the spinors associated with $H^\dagger$ in this case for an arbitary $V(x)$ because $H^\dagger$ and the operators $p_\pm$ do not have simaltaneous eigen functions as $[H^\dagger, p_\pm] \neq 0$. However for the special case when $V(x)$ is independent of x we can substitutes the operators $p_\pm$ by their eigenvalues as $[H^\dagger, p_\pm] = 0$. We therefore construct the positive definite $\eta$ for the special case

For this model,

$$H^\dagger = \left[\begin{array}{cc} m_0c^2 & c\mathbf{p_x} - V_0 \\ c\mathbf{p_x} + V_0 & -m_0c^2 \end{array}\right]. \tag{3.19}$$

Let $\left(\begin{array}{c} v_1 \\ v_2 \end{array}\right)$ be the two components spinor for $H^\dagger$ corresponding to the energy E, and satisfy,

$$H^\dagger \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right) = E \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right), \tag{3.20}$$

$|u_1>$ and $|u_2>$ are calculated as

$$|u_1 >= \left(\begin{array}{c} 1 \\ \frac{cp_x+V_0}{E+m_0c^2} \end{array}\right); \text{and } |u_2 >= \left(\begin{array}{c} \frac{-(cp_x-V_0)}{E+m_0c^2} \\ 1 \end{array}\right), \tag{3.21}$$

when $E$ is given by Eq. (3.12). Putting $|u_1>$ and $|u_2>$ from Eq. (3.21) in Eq. (3.14) we obtain

$$\eta_{II} = \begin{bmatrix} 1+(\frac{cp_x+V_0}{E-m_0c^2})^2- & \frac{2Ecp_x-2V_0m_0c^2}{E^2-m_0^2c^4} \\ \frac{2Ecp_x-2V_0m_0c^2}{E^2-m_0^2c^4} & 1+(\frac{cp_x-V_0}{(E+m_0c^2)})^2 \end{bmatrix} \tag{3.22}$$

We have constructed a positive definite $\eta$ for both the examples discussed in this paper. These positive definite $\eta$'s ensure the positive definite norms for all the state vectors associated with this theory and hence lead to a fullly consistent relativistic pseudo-Hermitian theory.

# 5 Conclusion

We have considered two completely different pseudo-Hermitian interactions in relativistic quantum mechanics. In the first example, we have dealt with Rashba interaction with imaginary coupling which plays an important role in studying spin Hall effect [49, 50, 51, 52]. This interaction is shown to be parity pseudo-Hermitian and the complete spectrum which we obtain by solving Dirac equation on the plane is real. It will be interesting to see whether the solutions of imaginary Rashba interaction lead to any new consequence in the study of spin Hall effect. Scalar pseudo-Hermitian interaction has been constructed with an arbitrary real parameter in 1+1 dimension and is shown that the spectrum is real by solving corresponding Dirac equation. Using spectral method we have further constructed a positive definite metric operator, $\eta$ for both the examples. Such positive definite $\eta$ ensures the positive definite norm for the state vectors. Thus we have a fully consistent quantum theory for the pseudo-Hermitian interactions in relativistic theory.

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